INTRODUCTION TO MEASUREMENTS AND UNCERTAINTY THEORY

1. Introduction.

Making a measurement means to ascertain the amount of a physical magnitude in terms of a standard unit. Therefore, it is essential to express the result of a measurement with a number and the appropiate units. For example, if somebody says that the speed of an airplane is 800 we are receiving an incomplete information. What do 800 mean? Meters per second? Kilometers per hour? Miles per hour?

Only when indicating the measured quantity along with its units, i.e. 800 km/h, useful information is being given. There are different systems of units, directly intertwined by arithmetic operations. Nowadays, the most used system of units is the Système International (SI), and this is the one that we will primarily use in the laboratory. The SI consists of seven base units: kilogram, meter, second, ampere, Kelvin, mol and candela.

Unfortunately, it is not possible to accomplish a measure with no uncertainty. These uncertainties can be due to multiple causes. Although some uncertainties are caused by human errors, some others are inherent to the process of measure and impossible to avoid. The order of magnitude of the total uncertainty in a measure represents its *accuracy.* When measuring a physical magnitude, we should know how reliable the measurement is. That *reliability* represents the measure precision, which can be known after estimating the uncertainty. In this guide we will learn how to estimate the uncertainty committed when accomplishing a measure in the laboratory.

2. Types of uncertainties.

There are three categories of uncertainties: Precision, systematic uncertainties and random uncertainties.

- **Precision:** Every measuring equipment have at least one scale. The smallest scale graduation determines how precisely one can infer the value of a parameter, i.e. the *accuracy* we can reach.

For example, a conventional ruler is divided into centimeters and millimeters. Therefore, any length that we measure using that ruler will have an uncertainty of about 1 mm. This uncertainty can only be decreased if we use a device with higher accuracy.

Then, the *Precision* correlates to the equipment accuracy, and it will be referred as ε_p in this guide.

- **Systematic errors (or determinate errors):** The most common types are instrumental error, operator error, and method error. Instrumental errors are usually due to a bad operation or miscalibration of the measuring instrument. They are often unidirectional, so they slant the result of the measurement consistently causing the value to be too large or too small always in the same amount. In principle, systematic errors can be eliminated if the nature of the bias is identified.

  An easy example of a systematic error is a losing time clock. The clock will always be slow from the right time, but the right time can be calculated if we know the clock delay. Another solution is to repair the clock!

  One way to avoid these errors is to check the correct operation and calibration of the measuring devices.

- **Random errors (or indeterminate errors):** They are the result of unavoidable sources of error shifting randomly the measured value to higher or lower values over the real one. The random errors, unlike the systematic errors, can be reduced, but never eliminated.

  For example, let us imagine that we want to time one minute using a digital clock which have a precision of a hundredth of a second. It can be seen that we cannot manage to reach a display of exactly 60 s. Our reaction time always produces an uncertainty of a few hundredths of a second. In this case the random error is only due to the operator skills, and not to the operation or calibration of the clock.

  The random error will be designated as ε_{acc}. in this guide. In next sections we will see the mathematical treatment of these uncertainties.

The total error of a measurement is usually a combination of the three types of errors described above. We don't know the true value of the total error (if we did it wouldn't be an error!), but it should be estimated as we are able to. The result of any physical measurement is expressed with a numerical value, x, and the associated degree of uncertainty, expressed as Δx.

3. Reporting the result of an experimental measurement.

Once we have measured a certain magnitude, x, and we know that its uncertainty is Δx, we should express the result as:

$$x = (x_0 \pm \Delta x) \qquad \text{[units]} \qquad (1)$$

the measured value and the uncertainty being expressed in the **same** units.

EXAMPLE

With a graduated ruler a person's height has been measured. The result is 1.76 m, and the uncertainty is 2 cm. The right way to express this measurement is:

$$Height = (1.76 \pm 0.02)\ m$$

4. Direct measurements.

A direct measurement is obtained using a measuring device. In this section we will study the uncertainties associated to this kind of measurements assuming that they are free from systematic errors.

4.1. Uncertainty in a single measurement of a magnitude.

When we perform a single measurement of a magnitude x, the uncertainty is just the instrument precision, $\Delta x = \varepsilon_p$. In order to determine ε_p we can define two cases depending on the type of instrument being *analogical* or *digital*.

- *Analogical instrument:* The precision is given as ½ the smallest scale graduation

$$\varepsilon_p = Smallest\ division \times \frac{1}{2}$$

For example, the precision of a ruler graduated in millimeters is $\varepsilon_p = 0.5\ mm$.

- *Digital instrument*: The precision is given as the smallest value that the device can measure.

$$\varepsilon_p = Smallest\ unit\ of\ the\ instrument$$

For example, the precision of a digital scale which can measure grams is $\varepsilon_p = 1\ g$.

4.2. Uncertainty in a finite set of measurements of a magnitude.

In general, a single measurement of a magnitude is not reliable enough, as many factors can drive it wrong, like a misreading of the scale, lack of attention when writing down the result, etc.

In order to avoid that, one can make repeated measurements of the same magnitude x. The result is a finite set of measurements $(x_1, x_2, \dots, x_n)$, each of them affected by the precision ε_p. But, which result is more accurate? Should we choose only one? In this case, we will get the best result by giving equal weight to all measurements, i.e., simply expressing the result as the average or mean, $\bar{x}$, over the set:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad (2)$$

The standard deviation of the measured values is represented by ε_{acc} and is given by the formula:

$$\varepsilon_{acc}(\bar{x}) = \sqrt{\frac{(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + \cdots + (x_n-\bar{x})^2}{n}} = \sqrt{\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}} \qquad (3)$$

The standard deviation is sometimes referred to as the *mean square deviation* and it measures how widely spread the measured values are on either side of the mean.

The final uncertainty, Δx, in our set of measurements will be the maximum value of either the precision and the standard deviation:

$$\Delta x = max\left(\varepsilon_p, \varepsilon_{acc}\right) \qquad (4)$$

Then, the most precise result will be $\bar{x}$ with an uncertainty Δx calculated from equation (4).

EXAMPLE:

15 length measurements from a bar were taken in the laboratory. The results, using a ruler graduated in millimeters, are listed below:

L (mm)	L (mm)	L (mm)
15.0	14,0	13.5
15.5	15.5	15.5
13.5	15.0	14.0
14.0	14.0	15.5
13.0	14.0	14.0

Let us try to get the best approximation to the true value of L *and estimate its uncertainty.*

As the ruler is an analogical device, the uncertainty in one single measurement is:

$$\Delta L_i = \varepsilon_p = \frac{1\,mm}{2} = 0.5\,mm$$

The best approximation to the true value of L is the mean of the set of measurement:

$$\bar{L} = \frac{\sum_{i=1}^{15} L_i}{15} = 14.4\,mm$$

In order to calculate the standard deviation on $\bar{L}$, it can be useful to use a table like the one below:

i	$L_i \pm 0.5\,mm$	$L_i - \bar{L}$	$(L_i - \bar{L})^2$
1	15.0	0.6	0.36
2	15.5	1.1	1.21
3	13.5	-0.9	0.81
4	14.0	-0.4	0.16
5	13.0	-1.4	1.96
6	14.0	- 0.4	0.16
7	15.5	1.1	1.21
8	15.0	0.6	0.36
9	14.0	-0.4	0.16
10	14.0	-0.4	0.16
11	13.5	-0.9	0.81
12	15.5	1.1	1.21
13	14.0	-0.4	0.16
14	15.5	1.1	1.21
15	14.0	-0.4	0.16
$\sum$	216		10.1

The standard deviation is calculated by means of equation (3).

$$\varepsilon_{acc}(\bar{L}) = \sqrt{\frac{\sum_{i=1}^{15}(L_i - \bar{L})^2}{15}} = 0.821\,mm$$

Then, the uncertainty of the measurement is:

$$\Delta L = max\left(\varepsilon_p, \varepsilon_{acc}\right) = max(0.5, 0.821) = 0.821\,mm$$

5. Significant figures: Rounding off.

When we estimate the uncertainty of a measurement, the result may present a large number of digits. There is no sense in overstating the precision of the measurement determination, as we just obtain an estimated value of the uncertainty and the first nonzero digit determines its magnitude.

In the last example, the uncertainty of the measurement was $\Delta L = 0.821\ mm$. The most significant figure indicates an uncertainty in the range of tenths of millimeter. With an uncertainty like this, it makes no difference to have an error of 1 thousandth a millimeter or 2 hundredths a millimeter.

The value of the uncertainty ΔL indicates that the measurement $\overline{L}$ is not reliable over tenths a millimeter. That way, it is established that the value of a measurement cannot have more precision than the uncertainty. The process consists in rounding off the uncertainty value in order to have only one nonzero digit.

The next criteria will be used to round off the value of a measurement and its uncertainty:

1. Both numbers should be expressed using the same units.

2. Only one non zero figure should be considered for the uncertainty. For example, an uncertainty of 0.345 s should be written as 0.3 s and an accuracy of 86 kg as 90 Kg. An exception is made when the most significant figure is 1 and the next figure is minor or equal to 5; in this case the number "1" remains, and the next figure remains too. For example, an error of 0.143 Kg rounds to 0.14 Kg.

In order to get these results we have rounded off the values according to the following rules:

a) If the first dropped digit is greater than or equal to 5, the last kept figure should be increased in 1 unit. For example, if we round off 0.861342 s, we should add one unit to the "8" and express this time period as 0.9 s, since the first dropped number is 6 > 5.

b) If the first dropped digit is minor than 5, the last kept figure remains the same. That way, 234.38 m rounded off in the most significant figure will be 200 m.

c) If the first dropped figure equals 5, we may find two situations:

- In the next dropped figures, there are nonzero values. In this case, last preserved figure grows in 1 unit. For example, 35.234 s is round to 40 s.

- All the drop figures are zero except a 5 number. In this case, the last kept figure remains the same. For example, 35.000 N rounds off 30N.

These round off rules are applicable both to the total error value and to the measurement value.

3. The measurement value should have the same precision than the uncertainty. Rounding off figures implies to turn into zero all the figures whose order of magnitude is minor than the uncertainty one. However, differing from the uncertainty, the result can have more than one nonzero digit. To round off the measurement value it is necessary to round off the uncertainty value before. For example, if the uncertainty is 0.7 Kg and the value of the measure is 25.784535 Kg, the final result is expressed as (25.8 ± 0.7) kg. If the uncertainty was 7 kg, the result would be (26 ± 7) Kg

Let us see the following example as a resume of the previous explanation.

EXAMPLE:

Round off and express these measurements and uncertainties according to the previous criteria.

Uncertainty (m)	Rounded uncertainty (m)	Measure (m)	Result (m)
0.018	0.02	0.987	0.99±0.02
0.068	0.07	25.8251	25.83±0.07
0.072	0.07	25.825	25.82±0.07
0.66	0.7	0.88	0.9±0.7
0.52	0.5	12	12.0±0.5
0.942	0.9	1.867	1.9±0.9
0.987	1	26.97	27±1
11.897	12	356.257	356±12
26	30	364	360±30
340	300	588.6	600±300
370.86	400	25.82	0±400

6. Absolute and relative errors: Analysis of the uncertainty.

So far we have only considered the **absolute error** Δx . In order to compare the error with a measurement itself, we can use the **relative error**, calculated by the following formula:

$$\varepsilon_{rel} = \frac{\Delta x}{x} \quad (5)$$

Or:

$$\varepsilon_{rel}(\%) = \frac{\Delta x}{x} \cdot 100 \quad (6)$$

If we express it in %:

EXAMPLE:

Suppose that we have measured the distance from the Earth to the Sun (R_{TS}) and from Marte to the Sun (R_{MS}), and that the obtained results are:

$R_{TS} = (1.5 \pm 0.4) \cdot 10^8\ km$

$R_{MS} = (22.8 \pm 0.4) \cdot 10^8\ km$

In both results, the absolute error is the same: $0.4 \cdot 10^8$ km. However, the uncertainty is much higher in the first case (R_{TS}) than in the second one (R_{MS}), as the relative error indicates:

$$\varepsilon_r(R_{TS}) = \frac{0.4 \cdot 10^8}{1.5 \cdot 10^8} \cdot 100 = 27\%$$

$$\varepsilon_r(R_{MS}) = \frac{0.4 \cdot 10^8}{22.8 \cdot 10^8} \cdot 100 = 2\%$$

7. Indirect measurements.

We may find in the laboratory physical magnitudes that cannot be directly measured but can be calculated from two or more directly measured quantities. In this case we are making **indirect measurements.** For example, an indirect measurement is the surface of a rectangle calculated from the measurement of its sides' lengths at the laboratory.

The uncertainty in the indirect measure derives from the uncertainties of the direct measurements used to calculate it. The method to calculate this uncertainty is known as propagation of uncertainty.

7.1. Propagation of uncertainty.

Suppose that we want to calculate the value of a magnitude y, dependent on a series of magnitudes $x_1, x_2, ..., x_n$, which can be known by a direct measurement in the laboratory:

$$y = f(x_1, x_2, ..., x_n) \tag{7}$$

First, we will calculate the mean and the uncertainty values $\bar{x}_i \pm \Delta x_i$. The best estimation of y is obtained by substituting in equation (3) the obtained values for $\bar{x}_i$.

$$\bar{y} = f(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \tag{8}$$

In order to estimate the uncertainty of Δx_i we can use the **differential analysis**, described below.

7.1.1. Differential analysis

Assuming that the uncertainty Δx_i of the variables x_i is small enough, it can be demonstrated that the uncertainty $\Delta \bar{y}$ can be calculated as:

$$\Delta \bar{y} = \left|\frac{\partial f}{\partial x_1}\right|_{\bar{x}_i} \Delta x_1 + \left|\frac{\partial f}{\partial x_2}\right|_{\bar{x}_i} \Delta x_2 + ... + \left|\frac{\partial f}{\partial x_n}\right|_{\bar{x}_i} \Delta x_n = \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}\right|_{\bar{x}_i} \Delta x_i \tag{9}$$

Be aware that all terms in equation (9) are absolute values and Δx_i is a positive number.

NOTE: $\frac{\partial f}{\partial x_i}$ is the **partial derivative** of *f* with respect to the variable x_i; i. e. it is the derivative of the function f with respect to x_i with the other variables held constant (see the examples). The subindex $\bar{x}_i$ indicates that this value should be evaluated in the result of the partial derivative.

EXAMPLE:

A physical magnitude y is determined by the equation: $y = a_1 x_1 - a_2 x_2$ where a_1 and a_2 are constants without uncertainty, and the uncertainties of x_1 and x_2 are Δx_1 and Δx_2 respectively. Calculate an equation for the uncertainty of y.

The partial derivatives are:

$$\frac{\partial y}{\partial x_1} = a_1 \text{ and } \quad \frac{\partial y}{\partial x_2} = -a_2$$

hence:

$$\Delta y = |a_1| \cdot \Delta x_1 + |a_2| \cdot \Delta x_2$$

EXAMPLE:

A physical magnitude y is determined by the equation: $y = x_1^n \cdot x_2^m$, where m and *n are constants without uncertainty, and the uncertainties of x_1 and x_2 are Δx_1 and Δx_2 respectively. Calculate an equation for the uncertainty of y.*

Solution:

The partial derivatives are:

$$\frac{\partial y}{\partial x_1} = n \cdot x_1^{n-1} \cdot x_2^m \quad \text{and} \quad \frac{\partial y}{\partial x_2} = m \cdot x_2^{m-1} \cdot x_1^n$$

hence:

$$\Delta y = |n \cdot x_1^{n-1} \cdot x_2^m| \cdot \Delta x_1 + \left|m \cdot x_2^{m-1} \cdot x_1^n\right| \cdot \Delta x_2$$

EXAMPLE:

Suppose that we have measured the diameter of a sphere with a precision of 1 cm: $D = 150 \pm 1\ cm.$ C*alculate the area (A) and the volume (V) of the sphere and the corresponding uncertainties.*

The surface and the volume of the sphere can be calculated as:

$$A = 4\pi R^2$$
$$V = \tfrac{4}{3}\pi R^3$$

The radius R is $R = \frac{D}{2} = 75\,\text{cm}$ and $\Delta R = \frac{\Delta D}{2} = 0.5\,\text{cm}$. Therefore, $R = 75.0 \pm 0.5\,\text{cm}$. For the area and the volume we will have:

$$A = 4\pi R^2 = 70685.8\,\text{cm}^2$$
$$\Delta A = \left|\frac{\partial A}{\partial R}\right| \Delta R = 8\pi R\,\Delta R = 942.5\,\text{cm}^2$$

$$V = \tfrac{4}{3}\pi R^3 = 1767145.9\,\text{cm}^3$$
$$\Delta V = \left|\frac{\partial V}{\partial R}\right| \Delta R = 4\pi R^2\,\Delta R = 35342.9\,\text{cm}^3$$

The final results will be:

$$A = 70700 \pm 900 = (70.7 \pm 0.9)10^3\ \text{cm}^2$$
$$V = 1770000 \pm 40000 = (17.7 \pm 0.4)10^5\ \text{cm}^3$$

EXAMPLE:

Two resistors, R_1 and R_2, have been measured 5 times each: R_1*=9.5, 9.8, 10.2, 9.9, 10.1 Ω (ohms); =15,5, 15,2, 14,8, 15,2, 15,0Ω. The precision of an individual measurement is 0.1Ω. Let us find the most likely value of R_1 and R_2 and estimate their corresponding uncertainties. If these resistors are connected in parallel, calculate the equivalent resistor of this circuit*.

We can resume the measurements of R_1 and R_2 in the following tables:

i	$R_{1i} \pm 0.1\Omega$	$R_{1i} - \overline{R_1}$	$(R_{1i} - \overline{R_1})^2$
1	9.5	- 0.4	0.16
2	9.8	- 0.1	0.01
3	10.2	0.3	0.09
4	9.9	0.0	0.00
5	10.1	0.2	0.04
$\sum$	49.5		0.30

i	$R_{2i} \pm 0.1\Omega$	$R_{2i} - \overline{R_2}$	$(R_{2i} - \overline{R_2})^2$
1	15.5	0.36	0.1296
2	15.2	0.06	0.0036
3	14.8	- 0.34	0.1156
4	15.2	0.06	0.0036
5	15.0	- 0.14	0.0196
$\sum$	75.7		0.272

The mean values of R_1 and R_2 are:

$$\overline{R_1} = \frac{\sum_{R_{1i}}}{5} = 9.9\,\Omega$$
$$\overline{R_2} = \frac{\sum_{R_{2i}}}{5} = 15.14\,\Omega$$

The standard deviations are:

$$\varepsilon_{acc}(\overline{R_1}) = \sqrt{\frac{\sum(R_{1i} - \overline{R_1})^2}{5}} = 0.24\Omega$$
$$\varepsilon_{acc}(\overline{R_2}) = \sqrt{\frac{\sum(R_{2i} - \overline{R_2})^2}{5}} = 0.23\Omega$$

The final results are:

$$\Delta\overline{R_1} = \max(\varepsilon_p(R_1), \varepsilon_{acc}(\overline{R_1})) = \max(0.1, 0.24) = 0.24\Omega$$
$$R_1 = 9.9 \pm 0.2\ \Omega$$

$$\Delta\overline{R_2} = \max(\varepsilon_p(R_2), \varepsilon_{acc}(\overline{R_2})) = \max(0.1, 0.23) = 0.23\Omega$$
$$R_2 = 15.1 \pm 0.2\ \Omega$$

For resistors in parallel, the equivalent resistor is:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

And the best value of R_{eq} is obtained substituting R_1 and R_2 for their corresponding $\overline{R_1}$ and $\overline{R_2}$ mean values, previously calculated:

$$\overline{R_{eq}} = \frac{\overline{R_1}\,\overline{R_2}}{\overline{R_1} + \overline{R_2}} = 5.986\Omega$$

The uncertainty of R_{eq} is:

$$\Delta\overline{R_{eq}} = \left|\frac{\partial R_{eq}}{\partial R_1}\right| \Delta\overline{R_1} + \left|\frac{\partial R_{eq}}{\partial R_2}\right| \Delta\overline{R_2}$$
$$= \frac{\overline{R_2}^2}{(\overline{R_1} + \overline{R_2})^2} \Delta\overline{R_1} + \frac{\overline{R_1}^2}{(\overline{R_1} + \overline{R_2})^2} \Delta\overline{R_2} = 0.104\Omega$$

Where mean values $\overline{R_1}$ and $\overline{R_2}$ have been substituted when evaluating the partial derivatives. The final result for R_{eq} is:

$R_{eq} = 6.0 \pm 0.1\,\Omega$.

8. Physical laws: Analysis of the dependence between variables.

Many physical laws establish the dependence of one variable in terms of another one. For example, the relation between the speed and the acceleration of an object affected by gravity is:

$$v(t) = v_0 + g \cdot t \qquad (10)$$

This equation establish a linear relation between v and t, where v_0 is the intercept and g is the slope.

A physical law is valid if the predicted relation between variables is experimentally verified. In the example, we should measure the speed $v(t_i)$ of an object at different sampling instants t_i and verify if the linear relation in equation (10) is true. If so, we can determine the values of v_0 and g from the experimental data.

The next two sections explain two methods to analyze the linear relations between two variables and to calculate the parameters of proportionality.

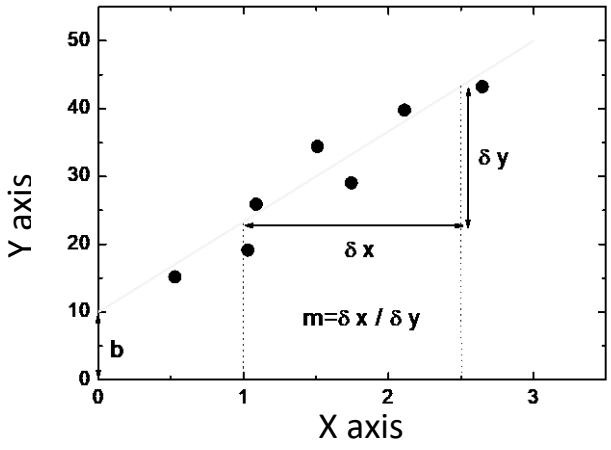


Figure 1. Linear fitting: m is the slope of the straight line and b is the intercept.

8.1. Graphical method.

When a variable y is linearly proportional to another variable x, the plot of a given set of pairs $\{(x_i, y_i)\}$ in a graph leads to a straight line. The general equation of a straight line is:

$$y = mx + b \tag{11}$$

Where *m* is the straight line's slope and b is the intercept.

Therefore, if the experimental data collected in the laboratory are supposed to obey a linear relation, when plotting them a spotted graph will be obtained, and a straight line can be drawn, as shown in Figure 1.

We can draw the "best" straight line through all the points, and we can calculate the numerical values of m and b as shown in Figure.1. It is important to see that the units for m and b are:

$$[m] = \frac{[y]}{[x]} \quad , \quad [b] = [y] \tag{12}$$

If a linear relation is not found when plotting the experimental data, it is evident that they do not obey a linear relation, and is not worth calculating *m* and *b*.

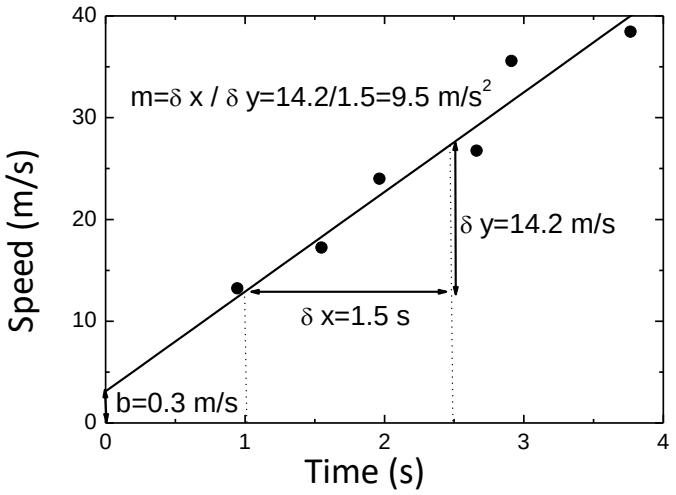


Figure 2. See the example in the text.

In the proposed example, the result of a set of measurements in the laboratory is a series of pairs $\{(v(t_i), t_i)\}$.

t (s)	$v(t)$ (m/s)
0.94	13.21
1.58	17.23
1.96	23.99
2.66	26.74
2.91	35.57
3.76	38.43

Figure 2 shows the data plot.

It can be observed that the experimental data spread as a straight line, as predicted by the physical law described in equation (10). Then, applying the described graphical method, g and v_0 values can be calculated as: $g = 9.5\ m/s^2$ and $v_0 = 3\ m/s$.

8.2. Least Squares Fitting

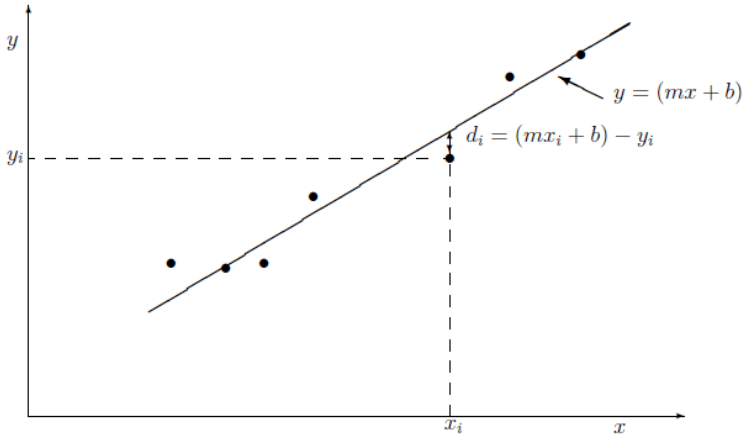
The graphical method is a rough way to fit data to a straight line and obtain its slope and intercept. It is easy, but not rigorous enough. These magnitudes can be obtained more accurately by using the **least squares method.**

Let us consider the equation of a straight line:

$$y = m\,x + b \tag{13}$$

The least squares method is a mathematical procedure for finding the best-fitting straight line to a given set of points by minimizing the sum of the squares of the distances from the experimental points to the straight line. If the experimental points are $(x_1, y_1), \cdots, (x_n, y_n)$, the best-fitting line will be found for values of m (the **slope of the straight line**) and b (the **intercept**) obeying the condition:

$$\sum (m \cdot x_i + b - y_i)^2 = minimum \tag{14}$$



It can be demonstrated that the values m and b obeying this condition are the solution of the following system of equations:

$$\begin{cases} \sum y_i = m\sum x_i + bn \\ \sum x_i y_i = m\sum x_i^2 + b\sum x_i \end{cases} \quad (15)$$

So the values of *m* and *b* are given by:

$$m = \frac{\begin{vmatrix} \sum y_i & n \\ \sum x_i y_i & \sum x_i \end{vmatrix}}{\begin{vmatrix} \sum x_i & n \\ \sum x_i^2 & \sum x_i \end{vmatrix}} = \frac{\sum y_i \sum x_i - n\sum x_i y_i}{(\sum x_i)^2 - n\sum x_i^2} \quad (16)$$

$$b = \frac{\begin{vmatrix} \sum x_i & \sum y_i \\ \sum x_i^2 & \sum x_i y_i \end{vmatrix}}{\begin{vmatrix} \sum x_i & n \\ \sum x_i^2 & \sum x_i \end{vmatrix}} = \frac{\sum x_i \sum x_i y_i - \sum y_i \sum x_i^2}{(\sum x_i)^2 - n\sum x_i^2} \quad (17)$$

m and b are obtained from the experimental data by applying (16) and (17), so they have uncertainties (Δm, Δb) associated to them. These uncertainties can be estimated by applying the following equations:

$$\Delta m = \sqrt{\frac{n\sigma^2}{n\sum x_i^2 - \left(\sum x_i\right)^2}} \quad (18)$$

$$\Delta b = \sqrt{\frac{\sigma^2 \sum x_i^2}{n\sum x_i^2 - \left(\sum x_i\right)^2}} \qquad (19)$$

These equations have been simplified by supposing that only the y_i values have uncertainties and that these uncertainties are the same for all data, that is: $\Delta(x_i) = 0 \quad \forall x_i$, and $\Delta(y_i) = \sigma \quad \forall y_i$. In the case that the uncertainties σ_{y_i} are not equal for all the y_i data, the σ value will be obtained as the average value of the data: $\sigma = \overline{\sigma}_y = \frac{1}{n}\sum_{i=1}^{n} \sigma_{y_i}$.

Before applying the least squares method, it is very advisable to plot the data in order to check that the relation between variables is approximately linear. *If that is not the case, and the relation between variables is not linear,* the method *is it NOT* applicable (see section 8.3).

When making a least-squares fitting, it is important to take into account all the decimal digits obtained during the intermediate steps of the process to calculate *m, b,* Δm and Δb . Only the final values should be rounded.

EXAMPLE:

Suppose that we measure several tension and intensity pairs of values at an electrical circuit in order to calculate the resistance of the circuit, and that the following values are obtained:

V (V)	I (A)
21 ± 1	20 ± 1
29 ± 1	30 ± 1
39 ± 1	40 ± 1
55 ± 2	50 ± 1
59 ± 2	60 ± 1

We know that $V = IR$. If we compare the equation $V = RI + b$ (being *b* = 0) with the equation of a straight line: $y = m\,x + b$ we deduct that *R* corresponds to the slope of the straight line, which can be fitted using the least squares method.

m and *b* can be obtained by applying equations (16) and (17). In order to do this, we have to calculate the following data:

$y = V$	$x = I$	xy	x^2
21	20	420	400
29	30	870	900
39	40	1560	1600
55	50	2750	2500
59	60	3540	3600
$\sum y_i = 203$	$\sum x_i = 200$	$\sum x_i y_i = 9140$	$\sum x_i^2 = 9000$

That lead to values of *m* and *b* of:

$$m = R = 1.02\Omega, b = -0.2V$$

The uncertainties of these values also have to be calculated. In order to do so, σ has to be obtained:

$$\sigma = \frac{1}{n}\sum_{i=1}^{n}\sigma_{y_i} = 1.4V$$

After using equations (19) and (20), the following results are obtained:

Δm = 0.0443 Ω

Δb = 1.878 V

So finally, the calculated resistance is: $m =$ R = (1.02±0.04) Ω

And the intercept is $b = (0 \pm 2)\ \mathrm{V}$. This value for b was the one expected from the theoretical equation $V = IR$.

8.3. Linearization of equations.

Both the graphical method and least squares method can be only applied when the relation between the two variables is linear. However, there are many physical laws described by nonlinear relations. For example, the equation relating the position of an object and the time in a motion with constant acceleration, provided that the initial velocity is zero, is given by:

$$s(t) = \frac{1}{2} a t^2 \qquad (20)$$

being *a* the acceleration of the object.

In some cases such as these it is possible to linearize this equation in order to obtain a straight line $y = mx + b$. There are two methods for linearizing an equation:

- **Changing variables**: We can perform a change of variables in order to obtain the equation of a straight line. In our example, if we rename the variables as:

$t^2 \rightarrow x \quad , \quad s \rightarrow y$

The resulting equation is:

$y = \frac{1}{2} a x$

By comparing with the equation of a straight line, we deduct that:

$m \rightarrow \frac{1}{2} a \quad , \quad b \rightarrow 0$

So $a = 2m$.

- **Using logarithms**: This method involves taking logarithms in the equation and using their properties to linearize the expression. In our example, if we take logarithms on the two sides of the equation:

$log(s) = log(\frac{1}{2} a t^2)$

and apply the properties of the logarithms, we obtain:

$log(s) = log(\frac{1}{2} a) + 2\, log(t)$

Our new variables are:

$x \rightarrow log(t) \quad , \quad y \rightarrow log(s)$

So we have a linear equation:

$y = log(\frac{1}{2} a) + 2x$

where:

$m \to 2$, $b \to log(\frac{1}{2}a)$

So finally: $a = 2\,e^{b}$

Once the equation has been linearized, a new table with the data of the new variables should be made. These data will be used to calculate m and b, and the plot of x vs. y should be approach to a straight line.

In our example and in the case of having used logarithms, we would construct the following table:

x	y
$log(t_1)$	$log(s_1)$
$log(t_2)$	$log(s_2)$
⋮	⋮
log(t_n)	$log(s_n)$

9. Graphs

9.1. General requirements.

One of the most useful ways of presenting experimental results is by plotting graphs. In order to do this, it is convenient to have in mind that:

- Experimental points should be clearly visible.

- Scales should be chosen so that data occupy the maximum space possible in the graph. The vertical and horizontal axes can be scaled different if it is necessary and the origin of the graph does not have to be $(0,0)$.

- Mark the axes at periodic intervals (and not for each experimental point). Use a comfortable scale to make calculations easy.

- When representing experimental points, add the uncertainty of each of them by drawing vertical and horizontal error bars. In order to do so, draw a cross centered on each experimental data. The arms of the cross should have a length of Δx (for the horizontal error bar) and Δy (for the vertical error bar).

- Add the units on each axis.

- If the experimental data can be fitted by a straight line, draw the line of best fit. This straight line is obtained by applying $y = m\,x + b$, being *m* and *b* the ones obtained from the least squares fitting.

9.2. Graph papers.

The use of millimeter paper (see figure 3a) makes it easier to draw a plot. When marking intervals on a millimeter paper, the distance between points is proportional to the difference between the values.

For example, if we represent 3 values being: $x_1 = 1$, $x_2 = 10$, and $x_3 = 100$ corresponding to points P_1, P_2, and P_3, the distance between P_1 and P_2 will be proportional to $x_2 - x_1 = 9$ and the distance between P_2 and P_3 will be proportional to $x_3 - x_2 = 90$.

There exists another type of paper, called *logarithmic* paper (see figure 3b). In this paper, the distance between points is proportional to the difference between the logarithms of the values. In a logarithmic graph, the distance between P_1 and P_2 will be proportional to $Log(x_2) - Log(x_1) = 1 - 0 = 1$, and the distance between P_3 and P_2 will be proportional to $Log(x_3) - Log(x_2) = 2 - 1 = 1$. In this way, the logarithmic paper makes the transformation to the logarithm scale, without having to calculate the actual logarithms. This represents an advantage in many cases. Representing $\log y$ vs. $\log x$ in millimeter paper is the same as representing *y* vs. x in logarithmic paper.

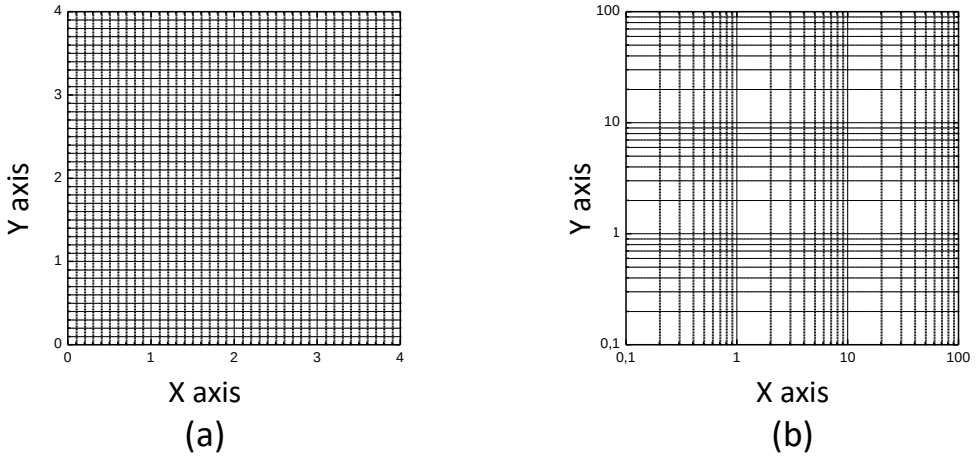


Figure 3. (a) Graph paper. (b) Logarithmic paper.

10. Bibliography.

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